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On dispersionless Lax representations for two-primary models

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Abstract
The equivalence between bi-Hamiltonian formulation and Lax representation of dispersionless integrable hierarchies associated with two-primary models of Frobenius manifolds is explicitly verified.

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1. Introduction

In a seminal work [8] (see also [9] for a review) Dubrovin explored the integrability of topological field theory from geometric point of view and reached to the notions called Frobenius manifolds. A Frobenius manifold is characterized by a function which can be viewed as the genus-zero free energy of the corresponding topological field theory. It turns out that for a finite-dimensional Frobenius manifold with good property one can associate it with a bi-Hamiltonian integrable hierarchy which is hydrodynamic type of Novikov and Dubrovin [10] and the free energy is a particular tau-function of the integrable hierarchy. For example, the free energy of the topological CP^1 model at genus zero defines a two-dimensional Frobenius manifold and the associated integrable hierarchy is the well-known dispersionless Toda (dToda) hierarchy [7, 9, 14]. At higher genus the free energy provides Gromov–Witten invariants and the bi-Hamiltonian hierarchy is a dispersive extension of the dToda hierarchy called extended Toda hierarchy [16, 25].

On the other hand, another important approach to dispersionless integrable hierarchies is the so-called Lax representation which not only provides a convenient way to construct conserved quantities but is also closely related to those concepts such as hodograph solutions, twistor construction, Hirota equations and Landau–Ginzburg formulation in topological field theories (see, e.g., [1, 4, 17, 18, 21, 22]). In this work we shall focus on dispersionless integrable hierarchies behind the two-dimensional Frobenius manifolds, including the dToda, Benney hierarchy (or dispersionless nonlinear Schrödinger equation) and dispersionless Dym hierarchy (dDym) (see, e.g., [19–21]). Although their Lax representations involving logarithmic functions have been proposed several years ago [5, 6, 14], however, only first few conserved densities and Lax equations have been demonstrated then for comparing with those results in bi-Hamiltonian formulation. In [15], Fairlie and Strachan provided a proof for the Lax representation of the dToda case by factorizing dispersionless Lax operators and representing the conserved charges in terms of Legendre polynomials. Recently, Carlet, Dubrovin and Zhang [2, 3, 13] provided the Lax representation of the extended Toda hierarchy by considering the logarithm of a difference operator and the Lax representation of the genus-zero CP^1 model can be recovered in the dispersionless limit.

In the following, we would like to give a direct proof of the equivalence between the dispersionless bi-Hamiltonian and Lax formulations of the whole set of conserved densities and hierarchy flows for the aforementioned three two-primary models in a unified fashion. We shall show that, due to the property of separation of variables in generating functions of conserved densities, the computations involving logarithmic function become much easier. Let us start from the dToda hierarchy and recall some notions in the context of Frobenius manifolds.

2. Dispersionless Toda hierarchy

2.1. Bi-Hamiltonian structure

The primary free energy of the topological CP^1 model is a two-primary solution of the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations [7, 24] of the form

$$F(t) = \tfrac{1}{2}(t^1)^2 t^2 + \mathrm{e}^{t^2}, \qquad t = (t^1, t^2), \tag{1}$$

which satisfies the quasi-homogeneity condition $\mathcal{L}_E F = 2F$ where the Euler vector $E(t) = t^1\partial_1 + 2\partial_2 \equiv E^\alpha\partial_\alpha$ and thus defines a two-dimensional Frobenius manifold [8, 9, 11, 12]. From (1) one can compute the structure constant $c_{\alpha\beta\gamma}(t) = \partial_\alpha\partial_\beta\partial_\gamma F_h(t)$ and the metric $\eta_{\alpha\beta} = c_{1\alpha\beta}$ which is used for lowering and raising indices. It turns out that $\eta_{11} = \eta_{22} = 0$, $\eta_{12} = \eta_{21} = 1$ and

$$c^1_{11} = c^2_{12} = c^2_{21} = 1, \qquad c^1_{22} = \mathrm{e}^{t^2}, \qquad c^\sigma_{\gamma\delta} = 0 \text{ otherwise},$$

where $c^\alpha_{\beta\gamma} = \eta^{\alpha\sigma}c_{\sigma\beta\gamma}$ and $\eta^{\alpha\beta} = (\eta_{\alpha\beta})^{-1}$. Since $\eta_{\alpha\beta}$ is a constant flat metric, we call the variables t^α the flat coordinates. Based on these data, one can introduce a genus-zero bi-Hamiltonian structure of the form

$$\begin{aligned}
\{t^\alpha(x), t^\beta(y)\}_1 &= \eta^{\alpha\beta}(t(x))\delta'(x-y) \equiv \mathcal{D}_1^{\alpha\beta}\delta(x-y),\\
\{t^\alpha(x), t^\beta(y)\}_2 &= \left(g^{\alpha\beta}(t(x))\partial_x + \Gamma^{\alpha\beta}_\gamma(t(x))t^\gamma_x\right)\delta(x-y) \equiv \mathcal{D}_2^{\alpha\beta}\delta(x-y),
\end{aligned} \tag{2}$$

which is of hydrodynamic type of Novikov and Dubrovin [10] (see also [23]). Here the metric $g^{\alpha\beta}$, or intersection form, is given by

$$g^{\alpha\beta}(t) = i_E(\mathrm{d}t^\alpha \cdot \mathrm{d}t^\beta) = E^\gamma c^{\alpha\beta}_\gamma = \begin{pmatrix} 2\,\mathrm{e}^{t^2} & t^1 \\ t^1 & 2 \end{pmatrix}$$

and $\Gamma^{\alpha\beta}_\gamma(t) = -g^{\alpha\sigma}\Gamma^\beta_{\sigma\gamma}$, the contravariant Levi-Civita connection of the flat metric $g^{\alpha\beta}$ is defined by

$$\Gamma^{\alpha\beta}_1(t) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma^{\alpha\beta}_2(t) = \begin{pmatrix} \mathrm{e}^{t^2} & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $\{\ ,\ \}_2 + \lambda\{\ ,\ \}_1$ forms a flat pencil of the corresponding Frobenius manifold and thus the hierarchy equations associated with the bi-Hamiltonian structure (2) are given by the commuting flows [9, 12]

$$\frac{\partial t^\alpha}{\partial T^{\beta,n}} = \{t^\alpha(x), H_{\beta,n}\}_1 = \{t^\alpha(x), \widetilde{H}_{\beta,n-1}\}_2, \qquad \alpha,\beta = 1,2; \qquad n \geqslant 0 \tag{3}$$

with

$$H_{\beta,n} = \int h_\beta^{(n+1)}(t(x))\,\mathrm{d}x, \qquad \beta = 1,2; \qquad n \geqslant -1, \qquad h_\alpha^{(0)} = \eta_{\alpha\beta}t^\beta,$$
$$\widetilde{H}_{\beta,n-1} = \frac{1}{n+\mu_\beta+1/2}\left(H_{\beta,n-1} - 2\delta_{1\beta}\frac{H_{2,n-2}}{(n+\mu_\beta+1/2)}\right),$$

where $\mu_1 = -1/2, \mu_2 = 1/2$. It may be noted [8] that the expression (3) of commuting flows fails to satisfy the second structure for $(\beta, n) = (1, 0)$ since this pair obeys the condition $n+\mu_\beta+1/2 = 0$. Hence the $T^{1,0}$ flow admits only the first Hamiltonian structure. A Frobenius manifold is resonant if it has such a pair (β, n). Since $\partial t^\alpha/\partial T^{1,0} = \partial t^\alpha/\partial x$ we may identify $T^{1,0} = x$. The first two nontrivial Hamiltonian equations of (3) are

$$\frac{\partial}{\partial T^{1,1}}\begin{pmatrix} t^1 \\ t^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(t^1)^2 + (t^2-1)\,\mathrm{e}^{t^2} \\ t^1t^2 \end{pmatrix}_x, \qquad \frac{\partial}{\partial T^{2,0}}\begin{pmatrix} t^1 \\ t^2 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{t^2} \\ t^1 \end{pmatrix}_x.$$

2.2. *Conserved densities*

The bi-Hamiltonian recursive relation (3) can be written in a form as $\partial_\alpha\partial_\beta h_\sigma(t,z) = zc_{\alpha\beta}^\gamma(t)\partial_\gamma h_\sigma(t,z)$ [8, 9] or in components

$$\partial_1^2 h_\alpha(t,z) = z\partial_1 h_\alpha(t,z) \tag{4}$$
$$\partial_1\partial_2 h_\alpha(t,z) = z\partial_2 h_\alpha(t,z) \tag{5}$$
$$\partial_2^2 h_\alpha(t,z) = z\,\mathrm{e}^{t^2}\partial_1 h_\alpha(t,z), \tag{6}$$

where $h_\alpha(t,z) = \sum_{n=0} h_\alpha^{(n)}(t)z^n$ are the generating functions of the conserved densities $h_\alpha^{(n)}$. Equations (4) and (5) together with normalization $\partial_1 h_\alpha(t,z) = zh_\alpha(t,z) + \eta_{1\alpha}$ yield $h_\alpha(t,z) = \mathrm{e}^{zt^1} f_\alpha(t^2,z) - \eta_{1\alpha}z^{-1}$, while (6) implies that $f_\alpha(t^2)$ satisfies $f_\alpha'' - z^2\mathrm{e}^{t^2} f_\alpha = 0$. Setting $y \equiv 2z\,\mathrm{e}^{t^2/2}$ then $v_\alpha(y) = f_\alpha(t^2)$ obeys

$$y^2v_\alpha'' + yv_\alpha' - y^2v_\alpha = 0, \tag{7}$$

which has a general solution of the form

$$v_\alpha(y) = C_\alpha^{(1)} I_0(y) + C_\alpha^{(2)} K_0(y),$$

where $C_\alpha^{(1)}$ and $C_\alpha^{(2)}$ are functions of z only, and I_0 and K_0 are the modified Bessel functions of first and second kinds of order zero, respectively. In view of the initial conditions $h_\alpha^{(0)}(t,0) = \eta_{\alpha\beta}t^\beta$ one has [12]

$$h_1(t,z) = -2\,\mathrm{e}^{zt^1}\{K_0(2z\,\mathrm{e}^{t^2/2}) + (\ln z + \gamma_E)I_0(2z\,\mathrm{e}^{t^2/2})\}$$
$$h_2(t,z) = z^{-1}\{\mathrm{e}^{zt^1} I_0(2z\,\mathrm{e}^{t^2/2}) - 1\}$$

or

$$h_1^{(n)}(t) = \sum_{s=0}^{[\frac{n}{2}]} \frac{(t^1)^{n-2s}\,\mathrm{e}^{st^2}(t^2-2c_s)}{(n-2s)!(s!)^2} \tag{8}$$

$$h_2^{(n)}(t) = \sum_{s=0}^{[\frac{n+1}{2}]} \frac{(t^1)^{n-2s+1}\,\mathrm{e}^{st^2}}{(n-2s+1)!(s!)^2}, \tag{9}$$

where the symbol $[x]$ denotes the largest integer less than x, $\gamma_E = 0.577\,215\,66\ldots$ is the Euler–Mascheroni constant and $c_s = \sum_{l=1}^{s} 1/l$ with $c_0 = 0$. The first few Hamiltonian densities are given by

$$h_1^{(0)} = t^2, \qquad h_1^{(1)} = t^1 t^2, \qquad h_1^{(2)} = \frac{(t^1)^2 t^2}{2} + (t^2 - 2)\,\mathrm{e}^{t^2},$$
$$h_2^{(0)} = t^1, \qquad h_2^{(1)} = \frac{(t^1)^2}{2} + \mathrm{e}^{t^2}, \qquad h_2^{(2)} = \frac{(t^1)^3}{6} + t^1 \mathrm{e}^{t^2}.$$

2.3. Lax representation

Proposition 1 [14]**.** *The conserved densities $h_\alpha^{(n)}(t)$ of the dToda hierarchy can be expressed in terms of the Lax function $L = p + t^1 + \mathrm{e}^{t^2} p^{-1}$ as*

$$h_1^{(n)}(t) = \frac{2}{n!}(L^n(\log L - c_n))_{[0]}, \qquad n \geqslant 0 \tag{10}$$

$$h_2^{(n)}(t) = \frac{1}{(n+1)!}(L^{n+1})_{[0]}, \qquad n \geqslant -1, \tag{11}$$

where $\left(\sum_k a_k p^k\right)_{[j]} = a_j$ and $\log L$ *is given by the prescription*

$$\log L = \frac{t^2}{2} + \frac{1}{2}\log\left(1 + t^1 p^{-1} + \mathrm{e}^{t^2} p^{-2}\right) + \frac{1}{2}\log\left(1 + t^1 \mathrm{e}^{-t^2} p + \mathrm{e}^{-t^2} p^2\right)$$

with the proviso that we shall Taylor expand the second term in p^{-1}, whereas in p for the last term.

Proof. For $h_1^{(n)}(t)$, by virtue of the 'scaling' property [14]

$$\frac{\mathrm{d}}{\mathrm{d}L} L^n(\log L - c_n) = nL^{n-1}(\log L - c_{n-1}),$$

we have $\partial h_1^{(n+1)}(t)/\partial t^1 = h_1^{(n)}(t), n \geqslant 0$ or $\partial_1 h_1(t,z) = z h_1(t,z)$. Then $h_1(t,z) = \mathrm{e}^{zt^1} f_1(t^2, x)$ where $f_1(t^2, z) = h_1(0, t^2, z) = \sum_{n=0}^{\infty} h_1^{(n)}(0, t^2) z^n$. Let us compute the quantities $h_1^{(n)}(0, t^2)$. By setting $t^1 = 0$ in (10), after some algebra, we get

$$h_1^{(2l+1)}(0, t^2) = 0, \qquad h_1^{(2l)}(0, t^2) = \frac{t^2 - 2d_l}{(l!)^2}, \qquad k \geqslant 0,$$

where

$$d_l = c_{2l} + (l!)^2 \sum_{k=0}^{l-1} \frac{(-1)^{l-k}}{k!(2l-k)!(l-k)}.$$

It turns out that

$$h_1(t,z) = \mathrm{e}^{zt^1} \sum_{l=0}^{\infty} h_1^{(2l)}(0, t^2) z^{2l} = \sum_{n=0}^{\infty} \left\{ \sum_{s=0}^{[\frac{n}{2}]} \frac{(t^1)^{n-2s}\,\mathrm{e}^{st^2}(t^2 - 2d_s)}{(n-2s)!(s!)^2} \right\} z^n.$$

Comparing with (8), the remaining task is to show that $d_s = c_s$ for $s \geqslant 0$. For $s = 0$, we have $d_0 = 0 = c_0$. On the other hand, using the identity

$$\sum_{l=0}^{m} (-1)^l \binom{n}{l} = (-1)^m \binom{n-1}{m}, \qquad n \geqslant 1, \qquad m \geqslant 0, \tag{12}$$

we have

$$\begin{aligned} d_{s+1} - d_s &= \frac{1}{2s+1} + \frac{1}{2s+2} + (s!)^2 \sum_{k=0}^{s} \frac{(-1)^{s-k}(k-s-1)}{k!(2s-k+2)!} \\ &= \frac{1}{2s+1} + \frac{1}{2s+2} - \frac{1}{(2s+1)(2s+2)} \\ &= \frac{1}{s+1} = c_{s+1} - c_s, \end{aligned}$$

which implies $d_s = c_s$ for $s \geqslant 0$. For $h_2^{(n)}(t)$, from (11) we have $\partial h_2^{(n+1)}(t)/\partial t^1 = h_2^{(n)}(t)$, $n \geqslant -1$ with $h_2^{(-1)} = 1$, or $\partial_1 h_2(t,z) = z h_2(t,z) + 1$. Then $h_2(t,z) = \mathrm{e}^{zt^1} f(t^2, x) - z^{-1}$ where $f_2(t^2, z) = h_2(0, t^2, z) + z^{-1} = \sum_{n=-1}^{\infty} h_2^{(n)}(0, t^2) z^n$. It is easy to show that

$$h_2^{(2l)}(0, t^2) = 0, \qquad h_2^{(2l-1)}(0, t^2) = \frac{\mathrm{e}^{lt^2}}{(l!)^2}, \qquad l \geqslant 0$$

and hence

$$h_2(t,z) = \mathrm{e}^{zt^1} \sum_{l=0}^{\infty} \frac{\mathrm{e}^{lt^2}}{(l!)^2} z^{2l-1} - z^{-1} = \sum_{n=0}^{\infty} \left\{ \sum_{l=0}^{[\frac{n+1}{2}]} \frac{(t^1)^{n-2l+1}\, \mathrm{e}^{lt^2}}{(n-2l+1)!(l!)^2} \right\} z^n,$$

which recovers the conserved densities (9). □

Proposition 2 [14]**.** *The Hamiltonian equations (3) of the dToda hierarchy can be written in the Lax form as*

$$\frac{\partial L}{\partial T^{\beta,n}} = \{B_{\beta,n}, L\}, \qquad \beta = 1, 2; \qquad n \geqslant 0 \tag{13}$$

with

$$B_{1,n} = \frac{2}{n!}(L^n(\log L - c_n))_{\geqslant 0}, \qquad B_{2,n} = \frac{1}{(n+1)!}(L^{n+1})_{\geqslant 0},$$

where the projection $\left(\sum_k a_k p^k\right)_{\geqslant j} = \sum_{k \geqslant j} a_k p^k$ *and the Poisson bracket* $\{f, g\} \equiv p\partial_p f \partial_x g - p\partial_x f \partial_p g$.

Proof. Let $B_{\beta,n} = \sum_{i \geqslant 0} b_{\beta,n}^{(i)}(t) p^i$, then from $\partial_1 B_{\beta,n}$ and $\partial_2 B_{\beta,n}$, the coefficients $b_{\beta,n}^{(i)}(t)$ satisfy

$$\frac{\partial b_{\beta,n}^{(i)}(t)}{\partial t^1} = b_{\beta,n-1}^{(i)}(t), \qquad \frac{\partial b_{\beta,n}^{(i)}(t)}{\partial t^2} = \mathrm{e}^{t^2} b_{\beta,n-1}^{(i+1)}(t), \qquad n \geqslant 0.$$

On the other hand, from the p^0 and p^{-1} terms of the Lax equation (13) we have

$$\frac{\partial t^1}{\partial T^{\beta,n}} = \left(\frac{\partial b_{\beta,n+1}^{(0)}(t)}{\partial t^2} \right)_x, \qquad \frac{\partial t^2}{\partial T^{\beta,n}} = \left(b_{\beta,n}^{(0)}(t) \right)_x,$$

which together with $b_{\beta,n}^{(0)}(t) = (B_{\beta,n})_{[0]} = h_\beta^{(n)}(t)$ implies the Hamiltonian flows (3). □

3. Benney and dDym hierarchies

Having established the Lax formulation for the dToda hierarchy, we now turn to another two-primary model with logarithmic-type primary free energy: the Benney hierarchy and the dDym hierarchy.

3.1. Benney hierarchy

The two-dimensional Frobenius manifold corresponding to the Benney hierarchy is described by the primary free energy

$$F(t) = \tfrac{1}{2}(t^1)^2 t^2 + \tfrac{1}{2}(t^2)^2\left(\log t^2 - \tfrac{3}{2}\right),$$

which satisfies $\mathcal{L}_E F = 4F$ with $E = t^1\partial_1 + 2t^2\partial_2$. Just like the dToda hierarchy, the associated bi-Hamiltonian structure can be constructed from the primary free energy as

$$\mathcal{D}_1^{\alpha\beta} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \qquad \mathcal{D}_2^{\alpha\beta} = \begin{pmatrix} 2\partial & t^1\partial + t_x^1 \\ t^1\partial & 2t^2\partial + t_x^2 \end{pmatrix}.$$

The Hamiltonian flows of the Benney hierarchy are defined as (3) with

$$\widetilde{H}_{\beta,n-1} = \frac{1}{n+\mu_\beta+1/2}\left(H_{\beta,n-1} - 2\delta_{2\beta}\frac{H_{1,n-2}}{(n+\mu_\beta+1/2)}\right),$$

where $\mu_1 = 1/2$, $\mu_2 = -1/2$ and the pair $(\beta, n) = (2, 0)$ is resonant. Following the similar discussions, the conserved densities of the Benney hierarchy can be expressed in terms of the modified Bessel functions as [12]

$$\begin{aligned} h_1(t,z) &= z^{-1}\sqrt{t^2}\,\mathrm{e}^{zt^1} I_1(2z\sqrt{t^2}) \\ h_2(t,z) &= z^{-1}\{2z\sqrt{t^2}\,\mathrm{e}^{zt^1}(K_1(2z\sqrt{t^2}) - (\log z + \gamma_E)I_1(2z\sqrt{t^2})) - 1\} \end{aligned}$$

or

$$h_1^{(n)}(t) = \sum_{s=0}^{[\frac{n}{2}]} \frac{(t^1)^{n-2s}(t^2)^{s+1}}{(n-2s)!s!(s+1)!} \tag{14}$$

$$h_2^{(n)}(t) = \frac{(t^1)^{n+1}}{(n+1)!} + \sum_{s=0}^{[\frac{n-1}{2}]} \frac{(t^1)^{n-2s-1}(t^2)^{s+1}}{(n-2s-1)!s!(s+1)!}(\log t^2 - c_{s+1} - c_s). \tag{15}$$

Proposition 3. *The conserved densities $h_\alpha^{(n)}(t)$ of the Benney hierarchy can be expressed in terms of the Lax function $L = p + t^1 + t^2 p^{-1}$ as*

$$h_1^{(n)}(t) = \frac{1}{(n+1)!}\operatorname{res}(L^{n+1}), \qquad n \geqslant -1 \tag{16}$$

$$h_2^{(n)}(t) = \frac{2}{n!}\operatorname{res}(L^n(\log L - c_n)), \qquad n \geqslant 0, \tag{17}$$

where $\operatorname{res}\left(\sum_k a_k p^k\right) = a_{-1}$ *and* $\log L$ *is given by the prescription*

$$\log L = \tfrac{1}{2}\log t^2 + \tfrac{1}{2}\log(1 + t^1 p^{-1} + t^2 p^{-2}) + \tfrac{1}{2}\log\left(1 + \tfrac{t^1}{t^2}p + \tfrac{1}{t^2}p^2\right)$$

with the proviso that we shall Taylor expand the second term in p^{-1}, whereas in p for the last term.

Proof. The proof is similar to that of the dToda case. From (16) and (17) we have $\partial_1 h_\alpha(t,z) = zh_\alpha(t,z) + \eta_{1\alpha}$ and thus $h_\alpha(t,z) = \mathrm{e}^{zt^1} f_\alpha(t^2,z) - \eta_{1\alpha} z^{-1}$ where $f_\alpha(t^2,z) = \sum_{k=0} h_\alpha^{(k)}(0,t^2) z^k + \eta_{1\alpha} z^{-1}$. By setting $t^1 = 0$ in (16) and (17), it is easy to show that

$$h_1^{(2l)}(0,t^2) = \frac{(t^2)^{l+1}}{l!(l+1)!}, \qquad h_1^{(2l-1)}(0,t^2) = 0$$

and

$$h_2^{(2l)}(0,t^2) = 0, \qquad h_2^{(2l-1)} = \frac{(t^2)^l}{l!(l-1)!}(\log t^2 - c_l - c_{l-1}), \qquad l \geqslant 1.$$

Substituting $h_\alpha^{(k)}(0,t^2)$ into $f_\alpha(t^2,z)$ we reach the conserved densities (14) and (15). □

Proposition 4. *The Hamiltonian equations of the Benney hierarchy can be written in the Lax form as*

$$\frac{\partial L}{\partial T^{\beta,n}} = \{B_{\beta,n}, L\}, \qquad \beta = 1,2; \qquad n \geqslant 0, \tag{18}$$

where

$$B_{1,n} = \frac{1}{(n+1)!}(L^{n+1})_{\geqslant 1}, \qquad B_{2,n} = \frac{2}{n!}(L^n(\log L - c_n))_{\geqslant 1}$$

and the Poisson bracket $\{f,g\} \equiv \partial_p f \partial_x g - \partial_x f \partial_p g$.

Proof. Let $B_{\beta,n} = \sum_{i\geqslant 0} b_{\beta,n}^{(i)}(t) p^{i+1}$ and consider the t^1 and t^2 derivatives over $B_{\beta,n}$ then we obtain the relations

$$\frac{\partial b_{\beta,n}^{(i)}(t)}{\partial t^1} = b_{\beta,n-1}^{(i)}(t), \qquad \frac{\partial b_{\beta,n}^{(i)}(t)}{\partial t^2} = b_{\beta,n-1}^{(i+1)}(t), \qquad n \geqslant 0.$$

With the help of the above relations, the p^0 and p^{-1} terms of the Lax equation (18) can be written as

$$\frac{\partial t^1}{\partial T^{\beta,n}} = \left(\frac{\partial\left(t^2 b_{\beta,n+1}^{(0)}(t)\right)}{\partial t^2}\right)_x, \qquad \frac{\partial t^2}{\partial T^{\beta,n}} = \left(\frac{\partial\left(t^2 b_{\beta,n+1}^{(0)}(t)\right)}{\partial t^1}\right)_x.$$

Thus the remaining task is to show that $t^2 b_{\beta,n}^{(0)}(t) = h_\beta^{(n)}(t)$. This is indeed the case since from the identities

$$\mathrm{res}(L^n \partial L/\partial p) = 0, \qquad \mathrm{res}(L^n(\log L - c_n)\partial L/\partial p) = 0, \qquad n \geqslant 0 \tag{19}$$

and $\partial L/\partial p = 1 - t^2 p^{-2}$ we have

$$\mathrm{res}(L^n) = t^2 (L^n)_{[1]}, \qquad \mathrm{res}(L^n(\log L - c_n)) = t^2 (L^n(\log L - c_n))_{[1]}.$$ □

3.2. dDym hierarchy

Finally, we come to the dDym hierarchy. The associated two-dimensional Frobenius manifold is described by the primary free energy

$$F(t) = \tfrac{1}{2}(t^1)^2 t^2 - \tfrac{1}{2}\log t^2,$$

which satisfies $\mathcal{L}_E F = 0$ with $E = t^1\partial_1 - 2t^2\partial_2$. The corresponding bi-Hamiltonian structure can be deduced from the primary free energy as

$$\mathcal{D}_1^{\alpha\beta} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \qquad \mathcal{D}_2^{\alpha\beta} = \begin{pmatrix} \frac{2}{(t^2)^2}\partial - \frac{2}{(t^2)^3} t_x^2 & t^1\partial - t_x^1 \\ t^1\partial + 2t_x^1 & -2t^2\partial - t_x^2 \end{pmatrix},$$

and the commuting Hamiltonian flows of the dDym hierarchy are defined as (3) with

$$\widetilde{H}_{\beta,n-1} = \frac{1}{n+\mu_\beta+1/2}\left(H_{\beta,n-1} + 2\delta_{1\beta}\frac{H_{2,n-4}}{(n+\mu_\beta+1/2)}\right),$$

where $\mu_1 = -3/2, \mu_2 = 3/2$ and the resonance occurs at the pair $(\beta, n) = (1, 1)$. The generating functions of conserved densities of the dDym hierarchy have the following exact forms [12]:

$$h_1(t,z) = z\sqrt{t^2}\,\mathrm{e}^{zt^1}\left\{2\left(\log z + \gamma_E - \frac{1}{2}\right)J_1\left(\frac{2z}{\sqrt{t^2}}\right) - \pi Y_1\left(\frac{2z}{\sqrt{t^2}}\right)\right\}$$

$$h_2(t,z) = z^{-1}\left\{z^{-1}\sqrt{t^2}\,\mathrm{e}^{zt^1}J_1\left(\frac{2z}{\sqrt{t^2}}\right) - 1\right\}$$

or

$$h_1^{(n)}(t) = \frac{(t^1)^n t^2}{(n)!} + \sum_{s=0}^{[\frac{n-2}{2}]}\frac{(-1)^s(t^1)^{n-2s-2}(t^2)^{-s}}{(n-2s-2)!s!(s+1)!}(\log t^2 + c_{s+1} + c_s - 1) \qquad (20)$$

$$h_2^{(n)}(t) = \sum_{s=0}^{[\frac{n+1}{2}]}\frac{(-1)^s(t^1)^{n-2s+1}(t^2)^{-s}}{(n-2s+1)!s!(s+1)!}, \qquad (21)$$

where J_1 and Y_1 are the Bessel functions of first and second kinds of order 1, respectively.

Proposition 5. *The conserved densities $h_\alpha^{(n)}(t)$ of the dDym hierarchy can be expressed in terms of the Lax function $L = -\frac{1}{t^2}p + t^1 + p^{-1}$ as*

$$h_1^{(n)}(t) = \begin{cases} -\mathrm{res}(L^{-1}), & n = 0 \\ -\frac{2}{(n-1)!}\,\mathrm{res}\left(L^{n-1}\left(\log L - c_{n-1} - \frac{\mathrm{i}\pi-1}{2}\right)\right), & n \geqslant 1 \end{cases} \qquad (22)$$

$$h_2^{(n)}(t) = \frac{1}{(n+2)!}\,\mathrm{res}(L^{n+2}), \qquad n \geqslant 0, \qquad (23)$$

where L^{-1} is expanded in p^{-1} and $\log L$ is given by the prescription

$$\log L = \frac{\mathrm{i}\pi}{2} - \frac{1}{2}\log t^2 + \frac{1}{2}\log(1 - t^1t^2p^{-1} - t^2p^{-2}) + \frac{1}{2}\log\left(1 + t^1p - \frac{1}{t^2}p^2\right)$$

with the proviso that we shall Taylor expand the second term in p^{-1}, whereas in p for the last term.

Proof. From (22) and (23) we have $\partial_1 h_\alpha(t,z) = zh_\alpha(t,z) + \eta_{1\alpha}$ and thus $h_\alpha(t,z) = \mathrm{e}^{zt^1}f_\alpha(t^2,z) - \eta_{1\alpha}z^{-1}$ where $f_\alpha(t^2,z) = \sum_{k=0}h_\alpha^{(k)}(0,t^2)z^k + \eta_{1\alpha}z^{-1}$. By setting $t^1 = 0$ in (22) and (23), it is easy to show that

$$h_1^{(0)} = t^2, \qquad h_1^{(2l)}(0,t^2) = \frac{(-1)^{l-1}(t^2)^{-l+1}}{l!(l-1)!}(\log t^2 + c_l + c_{l-1} - 1), \qquad h_1^{(2l-1)}(0,t^2) = 0$$

and

$$h_2^{(2l)}(0,t^2) = 0, \qquad h_2^{(2l-1)} = \frac{(-1)^l(t^2)^{-l}}{l!(l+1)!}, \qquad l \geqslant 1.$$

Substituting $h_\alpha^{(k)}(0,t^2)$ into $f_\alpha(t^2,z)$ we recover the conserved densities (20) and (21). □

Remark. We may redefine c_n as $c_n - c_{n-1} = 1/n$ with $c_0 = -1/2 + \mathrm{i}\pi/2$ then the logarithmic-type conserved densities in (22) are written as $\mathrm{res}(L^{n-1}(\log L - c_{n-1}))$. Here, however, we keep the definition of c_n same as the dToda case.

Proposition 6. *The Hamiltonian equations of the dDym hierarchy can be written in the Lax form as*

$$\frac{\partial L}{\partial T^{\beta,n}} = \{B_{\beta,n}, L\}, \qquad \beta = 1, 2; \qquad n \geqslant 0 \tag{24}$$

with

$$B_{1,n} = \begin{cases} -(L^{-1})_{\geqslant 2}, & n = 0 \\ -\frac{2}{(n-1)!}\left(L^{n-1}\left(\log L - c_{n-1} - \frac{\mathrm{i}\pi - 1}{2}\right)\right)_{\geqslant 2}, & n \geqslant 1, \end{cases}$$

$$B_{2,n} = \frac{1}{(n+2)!}(L^{n+2})_{\geqslant 2}, \qquad n \geqslant 0$$

where L^{-1} is expanded in p and the Poisson bracket $\{f, g\} \equiv \partial_p f \partial_x g - \partial_x f \partial_p g$.

Proof. Rewrite the La equation (24) as

$$\frac{\partial L}{\partial T^{\beta,n}} = \{L, \bar{B}_{\beta,n}\}$$

where

$$\bar{B}_{1,n} = \begin{cases} -(L^{-1})_{\leqslant 1}, & n = 0 \\ -\frac{2}{(n-1)!}\left(L^{n-1}\left(\log L - c_{n-1} - \frac{\mathrm{i}\pi - 1}{2}\right)\right)_{\leqslant 1}, & n \geqslant 1 \end{cases}$$

$$\bar{B}_{2,n} = \frac{1}{(n+2)!}(L^{n+2})_{\leqslant 1}, \qquad n \geqslant 0.$$

Let $\bar{B}_{\beta,n} = \sum_{i\geqslant 0} b^{(i)}_{\beta,n} p^{1-i}$ then we have

$$\frac{\partial b^{(i)}_{\beta,n}(t)}{\partial t^1} = b^{(i)}_{\beta,n-1}(t), \qquad \frac{\partial b^{(i)}_{\beta,n}(t)}{\partial t^2} = \frac{b^{(i+1)}_{\beta,n-1}(t)}{(t^2)^2}.$$

Extracting the p^0 and p^1 terms of the Lax equation (24) we have

$$\frac{\partial t^1}{\partial T^{\beta,n}} = -\left(\frac{\partial\left(t^2 b^{(0)}_{\beta,n+1}(t)\right)}{\partial t^2}\right)_x = \left(\frac{\partial h^{(n+1)}_\beta}{\partial t^2}\right)_x$$

$$\frac{\partial t^2}{\partial T^{\beta,n}} = -\left(\frac{\partial\left(t^2 b^{(0)}_{\beta,n+1}(t)\right)}{\partial t^1}\right)_x = \left(\frac{\partial h^{(n+1)}_\beta}{\partial t^1}\right)_x,$$

where we have used the fact that $h^{(n)}_\beta = -t^2 b^{(0)}_{\beta,n}$ by taking into account the similar identities (19) for the Lax operator $L = -\frac{1}{t^2}p + t^1 + p^{-1}$. □

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